Long-time asymptotics for some integrable nonlinear waves

Chueh-Hsin Chang
Center of Advanced Study in Theoretical Sciences

CASTS-LJLL Workshop on Applied Mathematics and Mathematical Sciences

May 26-29, 2014

- **Joint work with:**
- Prof. Tony W. H. Sheu
- Prof. Ching-Hao Yu
  Department of Ocean Science and Engineering,
  Zhejiang University, Zhejiang, People's Republic of China.
- Mr. Le Lin
  Department of Engineering Science and Ocean Engineering,
  National Taiwan University

Outline

- Shallow water waves, Camassa-Holm (CH) equation
- Long-time asymptotics of solutions of CH equation
- Direct / inverse scattering
- Numerical simulation of CH equation under a specified initial data
- Open problem & Conclusion

- Shallow water waves:
  the depth of the water is much smaller than the wavelength of the disturbance of the free surface.
- A tsunami can have a wavelength in excess of 100 km and period on the order of one hour. Because it has such a long wavelength, a tsunami is a shallow-water wave

- Since the early 1970s, it has been frequently assumed that solitary (or cnoidal) waves can be used to model some of the important features of tsunamis approaching the beach and shoreline, and that these theories, originating from the KdV equation, can define the proper input waves for physical or mathematical models of tsunamis (Madsen, Fuhrman & Schäffer (Journal of Geophysical research 2008))
- Deep water waves (rouge wave): occur far out at sea, and are greater than twice the size of surrounding waves, very unpredictable. Focusing nonlinear Schrödinger equation.

- John Scott Russell (1808-1882, a Scottish civil engineer): solitary wave
- In 1834, while conducting experiments to determine the most efficient design for canal boats, he discovered a phenomenon that he described as the wave of translation. In fluid dynamics the wave is now called Russell's solitary wave.
- Scott Russell spent some time making practical and theoretical investigations of these waves. He built wave tanks at his home and noticed some key properties.

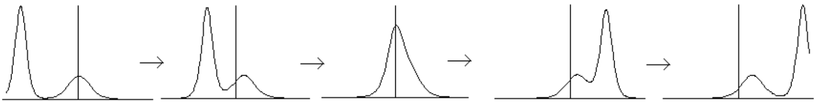
- Recreation of a solitary wave on the Scott Russell Aqueduct on the

Solitary wave approaching the beach in the lagoon of Molokai, Hawaii (photo by R.I. Odom)

- Many PDEs modeling shallow water waves corresponding to integrable systems.
- For example:
    - KdV equation:
      $u_t - 6uu_x + u_{xxx} = 0$
      (Korteweg and de-Vries (1895)),
    - BBM equation:
      $u_t - u_{xxt} + u_x + uu_x = 0$
      (Benjamin, Bona and Mahony (1972)),
    - Camassa-Holm equation
      $u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
      (Camassa, Holm (1993), Fokas & Fuchssteiner (1981)),
    - Degasperis-Procesi equation
      $u_t - u_{xxt} + 3\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}$
      (Degasperis, Procesi (1999))

- Periodic plane waves in shallow water, off the coast of Lima, Peru.

- What is the mechanism leading to the distinguished features of solutions ?
- Interaction of solitons: two traveling waves keep their shape and size after interaction



- Long time asymptotics: solutions behavior like solitons/oscillation waves/decay fastly in different space-time domain.
  In addition, $\exists$ transition regions (Painlevé transcendents).
  $\Longrightarrow$ **Direct / inverse scattering**.

- Linear special functions: Airy, Bessel, Whittaker, Lagurre, Hermite, hypergeometric function etc.....
- Nonlinear special functions: Painlevé transcendents
- Paul Painlevé: a French mathematician and politician. He served twice as Prime Minister of the Third Republic: 12 September – 13 November 1917 and 17 April – 22 November 1925. (Wikipedia)
- $w''(z) = F(w, w', z)$, F : rational in w, w', analytic in z.
  $\Rightarrow$Find F such that the singularities (except poles) of the solutions are "predictable", i.e., the singularities which depend on initial values are only ploes.

- Painlevé (1900, 1902), Gambier (1910):

$$
\begin{aligned}
&\text{(P1)} \quad w'' = 6w^2 + z,\\
&\text{(P2)} \quad w'' = 2w^3 + zw + \alpha,\\
&\text{(P3)} \quad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2+\beta}{z} + \gamma w^3 + \frac{\delta}{w},\\
&\text{(P4)} \quad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2\left(z^2-\alpha\right)w + \frac{\beta}{w},\\
&\text{(P5)} \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right)\\
&\qquad\qquad + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},\\
&\text{(P6)} \quad w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w'\\
&\qquad\qquad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \beta\frac{z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}.
\end{aligned}
$$

- How does nonlinear special functions (Painlevé transcendents) link with mathematical physics ?
  KdV equation (Ablowitz & Segur (Phys. D 1981))

Camassa-Holm equation

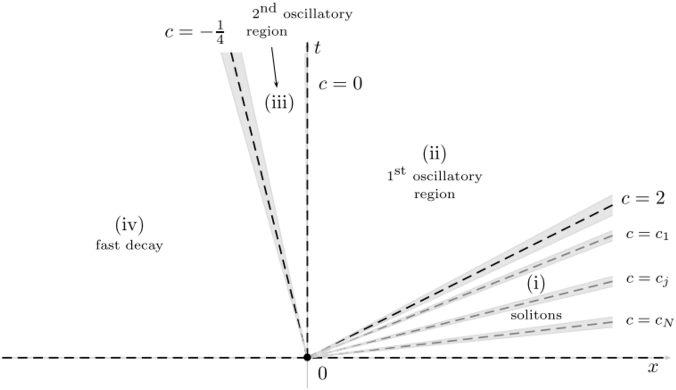
- Camassa-Holm (CH) equation from shallow water waves

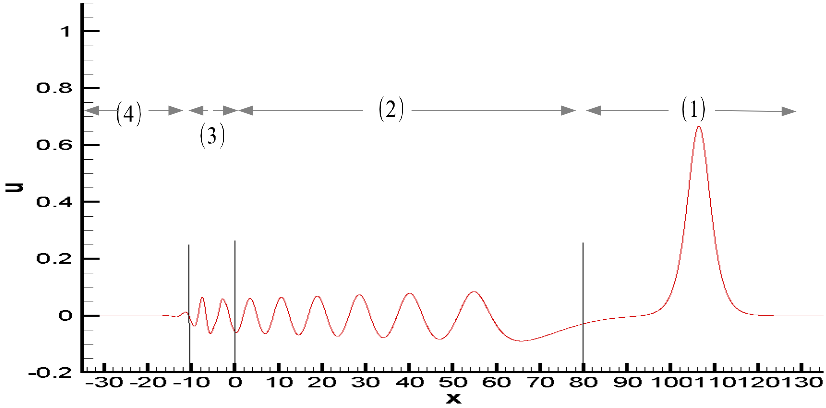
$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \;\; x \in \mathbb{R}, \;\; t > 0. \qquad \text{(CH)}$$

- $u = u(x,t)$: height of the water's free surface about a flat bottom.
- $\kappa > 0$: constant (critical shallow water speed).
- Camassa, Holm (1993), Fokas & Fuchssteiner (1981),
  Global existence (Constantin & Escher (1998)),
  periodic solution (Constantin & Mckean (1999)),
  wave breaking (Constantin (2000)),
  inverse scattering (Constantin (2000)),
  stability of solitary wave (Constantin & Strauss (2002)).

Long time asymptotics

- Boutet de Monvel, Kostenko, Shepelsky and Teschl (SIMA, 2009): Suppose $u(x,t)$ is a classical solution of CH equation, The asymptotics of solutions $u(x,t)$ of (CH) can be divided into four regions by considering the associated Riemann-Hilbert problem:




(4)
(3)
(2)
(1)
u
x
1
0.8
0.6
0.4
0.2
0
-0.2
-30 -20 -10 0 10 20 30 40 50 60 70 80 90 100 110 120 130

- Boutet de Monvel, Its and Shepelsky (SIMA, 2010):
- Continuing the results in 2009, in the two transition regions:
  (1) the region between 1st & 2nd sectors
  (2) the region between 3rd & 4th sectors
  the asymptotics of solutions is expressed by **second** Painlevé transcendents.

$$w''(z) = 2w^3(z) + zw(z).$$

- **Question**:
- "How long" is the time such that the solution close to these asymptotics ?
- How does the initial data $u(x, 0)$ influence the asymptotics form ?
- $\Rightarrow$ Construction of a specified $u(x, 0)$ such that the initial problem of CH equation can be numerically evoluted.

- **Direct and inverse scattering**
- Motivation: Fourier transform
- Example: $u_t + u_x + u_{xxx} = 0$

$$u(x,0) \xrightarrow{\text{direct Fourier transf.}} A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)e^{-ikx}\,dx$$

PDE

$$u_t + u_x + u_{xxx} = 0$$

time evolution

$$A(k)e^{-i\omega(k)t} \quad (\omega(k) = k - k^3)$$

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{-i\omega(k)t}e^{ikx}\,dx \xleftarrow{\text{inverse Fourier transf.}} A(k)e^{-i\omega(k)t}$$

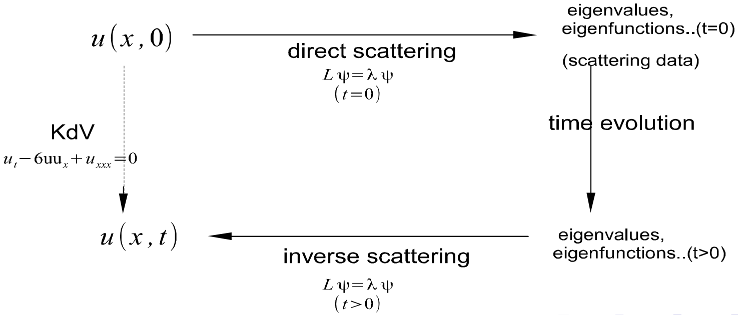
- KdV equation (Korteweg and de-Vries (1895)):

$$u_t - 6uu_x + u_{xxx} = 0, \;\; x \in \mathbb{R}, \;\; t > 0. \qquad \text{(KdV)}$$

$u = u(x,t)$: wave height above a flat bottom.

- The Lax pair of KdV is

$$L\psi = -\psi_{xx} + u(x,t)\,\psi = \lambda\psi,$$
$$P\psi = 2(2\lambda + u)\,\psi_x - u_x\psi, \text{ for } \psi \text{ satisfy } L\psi = \lambda\psi.$$



- **Scattering problem**:
  Given $u(x,t)$ $(t \geq 0)$, solve

$$L\psi = -\psi_{xx} + u(x,t)\,\psi = \lambda\psi$$

- Suppose

$$\int_{-\infty}^{\infty} (1+|x|)^{1+m} |u(x,t)|\, dx < \infty.$$

- By Faddeev (1958):
- (1) Discrete spectrum ($\lambda = k^2 < 0$):

  eigenvalues: $k = i\mu_j$, $j = 1, ..., N$ for some $N \in \mathbb{N}$, with corresponding eigenfunction: ψ_j,
  Let γ_j (normalization coefficients) be defined by

$$\psi_j(x) = \gamma_j e^{-\mu_j x} + o(1) \text{ as } x \to \infty$$

- (2) Continuous spectrum ($\lambda = k^2 > 0$) : $\hat{\psi} = \hat{\psi}(x, k)$: eigenfunction,

$$\hat{\psi} \sim \begin{cases} e^{-ikx} + R(k)\, e^{ikx}; \text{ as } x \to \infty, \\ T(k)\, e^{-ikx}; \text{ as } x \to -\infty. \end{cases}$$

  where $T(k)$: transmission coefficient, $R(k)$: reflection coefficient. $(|T(k)|^2 + |R(k)|^2 = 1.)$
- **Scattering data**: $R(k)$, μ_j, γ_j ($j = 1, ..., N$).

- By a series of inverse scattering theory, we have:
  - The number of discrete eigenvalues μ_j $(j = 1, ..., N)$ determine the number of solitary waves.
  - The appeareance of reflection coefficient $R(k)$ make the oscillation phenomenon occurs.
    If $R(k) \equiv 0 \Rightarrow$ pure soliton.

- Advantages of scattering problem compared with KdV equation:
- 1. Reduction of the differential order:
  2nd order derivative in $L\psi = -\psi_{xx} + u(x,t)\psi$ / 3rd order derivative in KdV equation.
- 2. (Lax1) is linear / KdV equation is nonlinear.
- 3. In $L\psi = -\psi_{xx} + u(x,t)\psi$, t is only a parameter.
- 4. Physical meaning to explain the appearence of soliton and long time asymptotic behaviors for solutions.

- **KdV results**
- Faddeev (1958), Zabusky and Kruskal (PRL, 1965).
- Gardner, Greene, Kruskal and Miura (PRL, 1967).
- Lax (CPAM 1968, 1975)
- Deift and Trubowitz (CPAM 1979), Marchenko (1986), Beals, Deift & Tomei (1988).
- *n*-solitons:
  Hirota (1971), Tanaka (1972, 73), Wadati & Toda (1972).
- Asymptotic behaviors:
  Zakharov & Manakov (1976), Ablowitz & Segur (1977),Segur & Ablowitz (1981), Deift, Its & Zhou (1993), Deift & Zhou (1993), Deift, Venakides & Zhou (1994), Grunert & Teschl (2009), et al...

- CH equation $u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
- Let the momentum $w(x,t) := u(x,t) - u_{xx}(x,t) + \kappa$.
- Cauchy problem for CH equation:
- We find the initial condition $u(x,0)$ s.t. $w(x,0)$ satisfy:
  (i) $w(x,0) > 0$, $u(x,0)$: smooth, rapidly decreasing as $|x| \to \infty$,

  $$\text{(ii)} \int_{\mathbb{R}} (1+|x|)^{1+m} \left(|w(x,0)-\kappa| + |w_x(x,0)| + |w_{xx}(x,0)|\right) dx < \infty \tag{MIC}$$

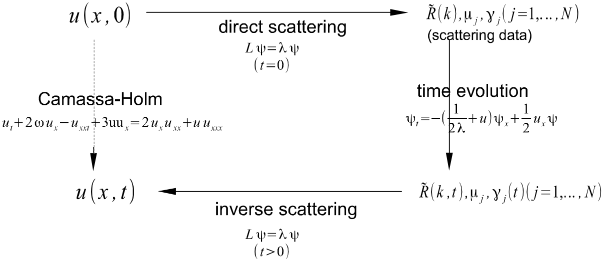
  for some $m \in \mathbb{N}$.
- Existence of classical solutions: Constantin and Escher (1998).
- Constantin (2001): $w(x,t) > 0$ for all $t > 0$.

- The Lax pair of CH (Camassa and Holm (1993)) is

$$L\psi = \frac{1}{w}\left(-\psi_{xx} + \frac{1}{4}\psi\right), \tag{Lax1}$$

$$\begin{aligned} P\psi = -\left(\tfrac{1}{2\lambda} + u\right)\psi_x + \tfrac{1}{2}u_x\psi, \\ \text{for } \psi \text{ satisfy } L\psi = \lambda\psi. \end{aligned} \tag{Lax2}$$

- $(\psi_t)_{xx} = (\psi_{xx})_t$ iff CH equation holds.



$$L\psi = \frac{1}{w(x,t)}\left(-\psi_{xx} + \frac{1}{4}\psi\right) = \lambda\psi \qquad \text{(Lax1)}$$

- Let

$$\lambda = \frac{1}{\kappa}\left(\frac{1}{4} + k^2\right), \ \tilde{\psi}(y) = \left(\frac{w(x,t)}{\kappa}\right)^{\frac{1}{4}}\psi(x),$$
$$y = x - \int_x^\infty \left(\sqrt{\frac{w(r,t)}{\kappa}} - 1\right) dr$$

(Liouville transform), then (Lax1) can be transformed to

$$-\tilde{\psi}_{yy} + q(y,t)\,\tilde{\psi} = k^2\tilde{\psi}$$

with

$$q(y,t) = \frac{w_{yy}(y,t)}{4w(y,t)} - \frac{3}{16}\frac{(w_y)^2(y,t)}{w^2(y,t)} + \frac{\kappa - w(y,t)}{4w(y,t)}.$$

- By (MIC) $\Longrightarrow$

$$\int_{-\infty}^{\infty}(1+|y|)^{1+m}\,|q(y,0)|\,dy < \infty.$$

$\Longrightarrow$ Scattering data: $\tilde{R}(k)$, κ_j, γ_j $(j = 1, \ldots, N)$ w.r.t. $q(y,0)$

- Constantin (Proc. R. Soc. Lond. 2001): continuous spectrum,
- Johnson (JFM, 2002), Lenells (JNMP, 2002),
  Constantin and Lenells (JNMP, 2003), (Phys. Lett-A, 2003)
  Yishen Li and Jin E. Zhang (Proc. R. Soc. Lond. 2004),
  Constantin, Gerdjikov & Ivanov (IP, 2007) et al...
- Only inverse scattering and the scattering data are assumed to be known, $+$ reflectionless (only discrete spectrum exist)
  $\Longrightarrow$ construct the pure n- soliton solutions ($n \in \mathbb{N}$).

- Two-soliton:

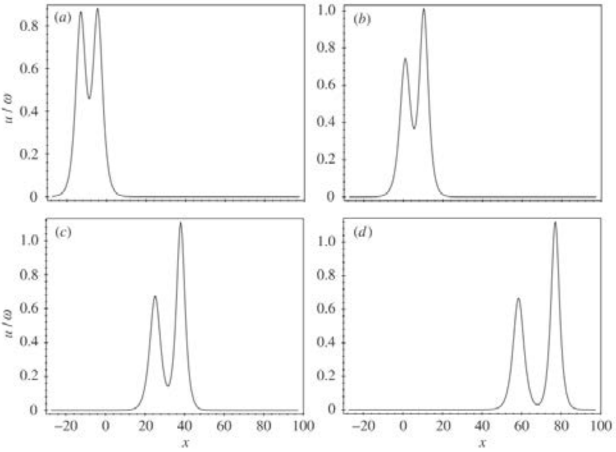


Figure 3. A two-soliton solution, u/ω against x, for $k_1 = 0.6$, $k_2 = 0.5$, $\alpha_1 = \alpha_2 = 0$, at times $T = 2\omega t$: (a) -3, (b) 7, (c) 25, (d) 50.

- Three-soliton:

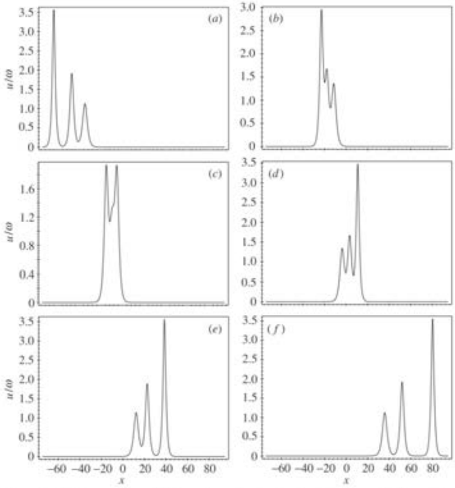
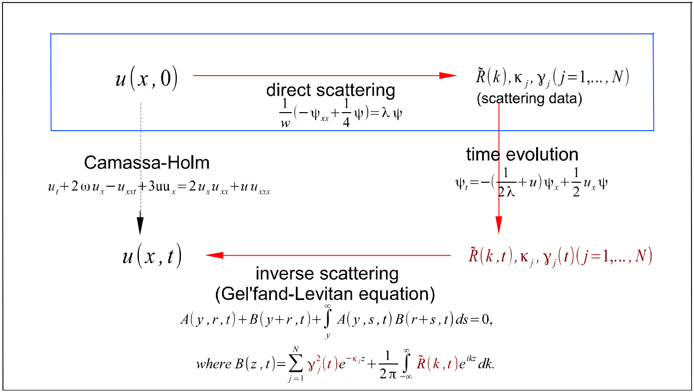


Figure 6. A three-soliton solution, u/ω against x, for $k_1 = 0.6$, $k_2 = 0.7$, $k_3 = 0.8$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, at times $T = 2\omega t$: (a) −20, (b) −5, (c) −1.5, (d) 5, (e) 15, (f) 30.

- Direct scattering problem:



- Given a scattering data, what is the corresponding initial condition $u(x,0)$?
- To the author's knowledge, non reflectionless case i.e., the continuous spectrum is non empty, has not been explored.

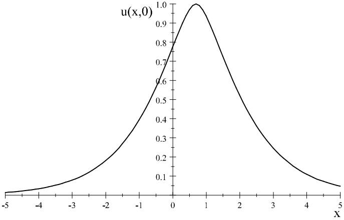
- **Theorem**:
Let $0 < q_0 < 1$, consider the CH equation subject to the following initial condition

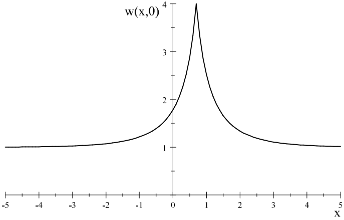
$$u(x,0) = \begin{cases} \frac{\kappa A(A+1+\log(e^x - A))}{e^x}, & \text{for } x \geq \log(1+A), \\ \frac{\kappa A\left(A+1+\log\left((1+A)^2 e^{-x} - A\right)\right)}{(1+A)^2 e^{-x}}, & \text{for } x < \log(1+A). \end{cases}$$

where $A := \frac{q_0}{1-q_0}$. The above initial condition in space-time domain corresponds to the following scattering data in spectral domain:

$$R(k) = \frac{-q_0}{q_0 + 2ik}, \quad \mu_1 = \frac{q_0}{2}, \quad \gamma_1 = \sqrt{\frac{q_0}{2}}.$$

- Momentum $w(x,t) := u(x,t) - u_{xx}(x,t) + \kappa$.
- Initial momeutum
$$w(x,0) = \begin{cases} \kappa\left(\frac{1}{1-Ae^{-x}}\right)^2, & \text{for } x \geq \ln(1+A), \\ \kappa\left(\frac{(1+A)^2}{(1+A)^2 - Ae^{x}}\right)^2, & \text{for } x < \ln(1+A). \end{cases}$$
$\Longrightarrow w(x,0) > 0$.
- Consider $q_0 = \frac{1}{2}$, $\kappa = 1$:





- At $t = 0$,

$$
\begin{aligned}
y &= x - \int_x^\infty \left(\sqrt{\frac{w(r,0)}{\kappa}} - 1 \right) dr \\
&= \begin{cases} \ln\left(e^x - A\right), & \text{for } x \geq \ln\left(1 + A\right), \\ -\ln\left(\left(1 + A\right)^2 e^{-x} - A\right), & \text{for } x < \ln\left(1 + A\right). \end{cases}
\end{aligned}
$$

and $-\tilde{\psi}_{yy} + q(y,0)\,\tilde{\psi} = k^2\tilde{\psi}$ is

$$
-\tilde{\psi}_{yy} - q_0\delta(y)\,\tilde{\psi} = k^2\tilde{\psi}.
$$

- Drazin and Johnson (1989).

- (1) **Soliton region**: $c := \frac{x}{t} > 2 + C$ $\forall$ small $C > 0$.
  Let $c_j := \frac{1}{2\left(\frac{1}{4}-\mu_j\right)}$, $(j = 1, ..., N)$, $\varepsilon > 0$ small s.t. the intervals $[c_j - \varepsilon, c_j + \varepsilon]$ are disjonit & $[c_j - \varepsilon, c_j + \varepsilon] \subset (2, \infty)$ $\forall j$.
  If $\left|\frac{x}{t} - c_j\right| < \varepsilon$ for some j :
  $$u(x,t) \sim \frac{32\kappa\mu_j^2}{\left(1-4\mu_j^2\right)^2} \frac{\alpha\left(y\left(x-\kappa c_j t-\xi_j\right)\right)}{\left(1+\alpha\left(y\left(x-\kappa c_j t-\xi_j\right)\right)\right)^2+\frac{16\mu_j^2}{1-4\mu_j^2}\alpha\left(y\left(x-\kappa c_j t-\xi_j\right)\right)}.$$
  where
  $$\alpha(y) = \frac{\hat{\gamma}_j^2}{2\mu_j} e^{-2\mu_j y},\ x = y + \log \frac{1+\alpha(y)\frac{1+2\mu_j}{1-2\mu_j}}{1+\alpha(y)\frac{1-2\mu_j}{1+2\mu_j}},$$
  $$\hat{\gamma}_j = \gamma_j \Pi_{i=j+1}^N \frac{\mu_i-\mu_j}{\mu_i+\mu_j},\ \xi_j = 2\sum_{i=j+1}^N \log \frac{1+2\mu_i}{1-2\mu_i}.$$
  If $\left|\frac{x}{t} - c_j\right| \geq \varepsilon$ for all j : $u(x,t) \sim 0$.

- (2) **First oscillatory region**: $0 \leq c := \frac{x}{t} > 2 - C$.

$$u(x,t) \sim \frac{c_1^{(0)}}{\sqrt{t}} \sin\left(c_2^{(0)} t + c_3^{(0)} \log t + c_4^{(0)}\right)$$

  where $c_m^{(0)}$ $(m = 1, ..., 4)$ are functions of c, which are determined by $u_0(x)$ in terms of the corresponding $R(k)$.
  In particular, $c_1^{(0)} = -c_5(c)\sqrt{\frac{-1}{2\pi}\log\left(1 - R(c_6(c))\right)^2}$
  where $c_5(c)$, $c_6(c)$ are functions of c.

- (3) **Second oscillatory region**: $\frac{-1}{4} + C < c = \frac{x}{t} < 0$.

$$u(x,t) \sim \sum_{j=0}^{1} \frac{c_1^{(j)}}{\sqrt{t}} \sin\left(c_2^{(j)} t + c_3^{(j)} \log t + c_4^{(j)}\right)$$

  where $c_m^{(j)}$ $(m = 1, ..., 4,\ j = 0, 1)$ are functions of c with $c_1^{(j)}$ have the forms similar to $c_1^{(0)}$.

- (4) **Fast decay region**: $c = \frac{x}{t} < \frac{-1}{4} - C$.
  $u(x,t) \sim 0$.

- For $\left|\frac{x}{t}-2\right| t^{\frac{2}{3}} < C$ with any $C > 0$,

$$u(x,t) = -\left(\frac{4}{3}\right)^{\frac{2}{3}} \frac{1}{t^{\frac{2}{3}}} \left(w^2(z) - w'(z)\right) + O\left(t^{-1}\right),$$

where $z = 6^{\frac{-1}{3}}\left(\frac{x}{t}-2\right) t^{\frac{2}{3}}$,

- For $\left|\frac{x}{t}+\frac{1}{4}\right| t^{\frac{2}{3}} < C$ with any $C > 0$,

$$u(x,t) = \frac{12^{\frac{1}{6}}}{t^{\frac{1}{3}}} w_1(z_1) \sin\left(\frac{-3\sqrt{3}}{4} t - \frac{3^{\frac{5}{6}}}{2^{\frac{4}{3}}} z_1 t^{\frac{1}{3}} + \Delta\right) + O\left(t^{-\frac{2}{3}}\right),$$

where $z_1 = -\left(\frac{16}{3}\right)^{\frac{1}{3}}\left(\frac{x}{t}+\frac{1}{4}\right) t^{\frac{2}{3}}$, Δ as a function depends on $u(x,0)$.
$w(z)$ & $w_1(z)$: the real-valued, non-singular solution of (P2) equation

$$w''(z) = 2w^3(z) + zw(z). \tag{P-II}$$

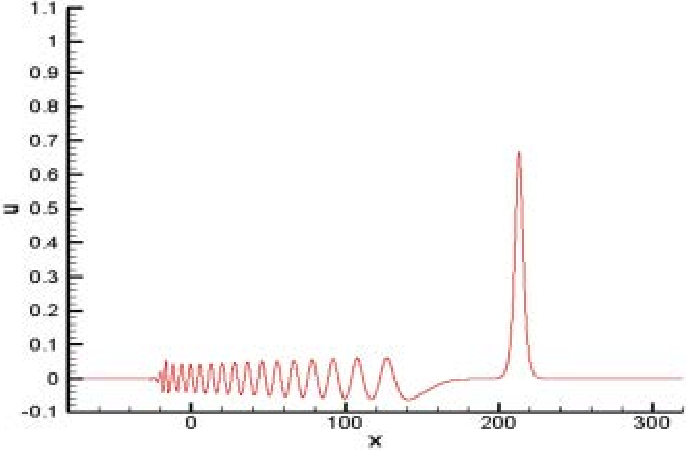
with $w(z) \sim -R(0)\mathrm{Ai}(z)$ as $z \to \infty$, $w_1(z) \sim \left|R\left(\frac{\sqrt{3}}{2}\right)\right| \mathrm{Ai}(z)$ as $z \to \infty$, $\left(\left|R\left(\frac{\sqrt{3}}{2}\right)\right| < 1\right)$.

Numerical simulation

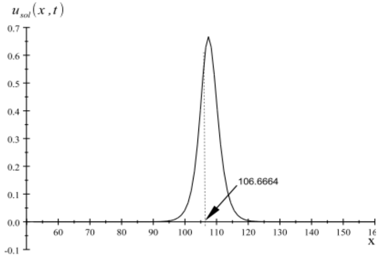
- Solve CH equation with

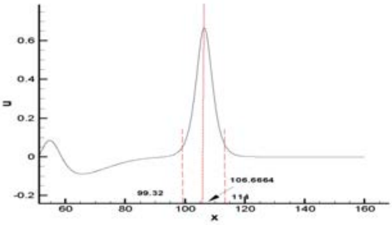
$$u(x,0) = \begin{cases} \frac{\kappa A(A+1+\log(e^x - A))}{e^x}, & \text{for } x \geq \log(1+A), \\ \frac{\kappa A\left(A+1+\log\left((1+A)^2 e^{-x} - A\right)\right)}{(1+A)^2 e^{-x}}, & \text{for } x < \log(1+A). \end{cases}$$

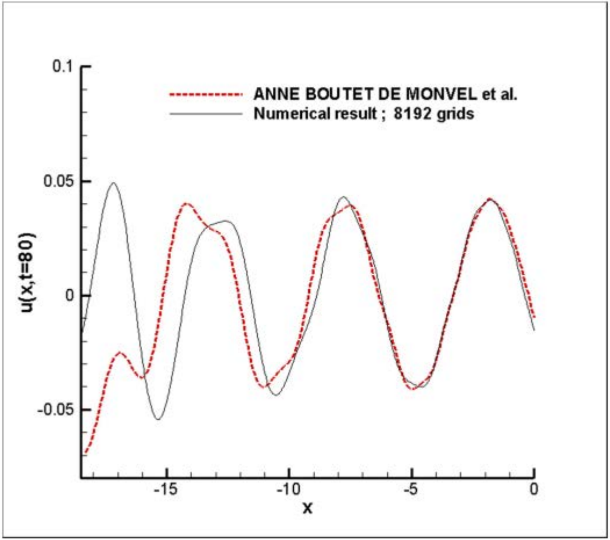
- $t = 80$:

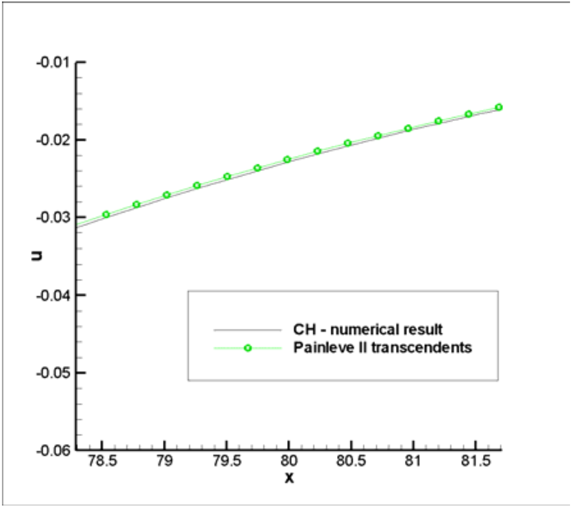


- $t = 40$:


$u_{sol}(x,t)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
60
70
80
90
100
110
120
130
140
150
16
X
106.6664


0.6
0.4
0.2
0
-0.2
u
60
80
100
120
140
160
x
99.32
106.6664


ANNE BOUTET DE MONVEL et al.
Numerical result ; 8192 grids
0.1
0.05
0
-0.05
u(x,t=80)
-15
-10
-5
0
x


-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
u
78.5
79
79.5
80
80.5
81
81.5
X
CH - numerical result
Painleve II transcendents

- Idea to find $u(x,0)$: Constantin's approach:
- Constantin (2001 RSPA):
- Find the positive solution $C = C(y,t)$ from

$$C_{yy} = C\left(q(y,t) + \frac{1}{4}\right) - \frac{\kappa}{4C^3}, \quad \lim_{|y|\to\infty} C(y,t) = \kappa^{\frac{1}{4}}.$$

  then $w(y,t) = C^4(y,t)$.
- Find change of variable between y and x by solving

$$\frac{dy}{dx} = \left(\frac{w(y,t)}{\kappa}\right)^{\frac{1}{2}}, \quad \lim_{x\to\infty}(y(x) - x) = 0.$$

- Let the solution be $y = \hat{y}(x,t)$.
- $w(x,t) = w(\hat{y}(x,t), t)$.

- Find the positive solution $C = C(y,0)$ from

$$C_{yy} = C\left(q(y,0) + \frac{1}{4}\right) - \frac{\kappa}{4C^3}, \lim_{|y|\to\infty} C(y,t) = \kappa^{\frac{1}{4}}$$

  with $q(y,0) = q(y,0) = -q_0\delta(y)$.
  then $w(y,0) = C^4(y,0) = \kappa\left(Ae^{-|y|} + 1\right)^2$.
- Find change of variable between y and x by solving

$$\frac{dy}{dx} = \left(\frac{w(y,0)}{\omega}\right)^{\frac{1}{2}} 1 + Ae^{-|y|}, \lim_{x\to\infty}(y(x) - x) = 0.$$

- The solution is

$$y = \begin{cases} \log(e^x - A), & \text{for } x \geq \log(1+A), \\ -\log\left((1+A)^2 e^{-x} - A\right), & \text{for } x < \log(1+A). \end{cases}$$

- Solve $w = u - u_{xx} + \kappa$, i.e., $\frac{w}{\kappa}u_{yy} + \frac{1}{2\kappa}w_y u_y - u = \kappa - w$.

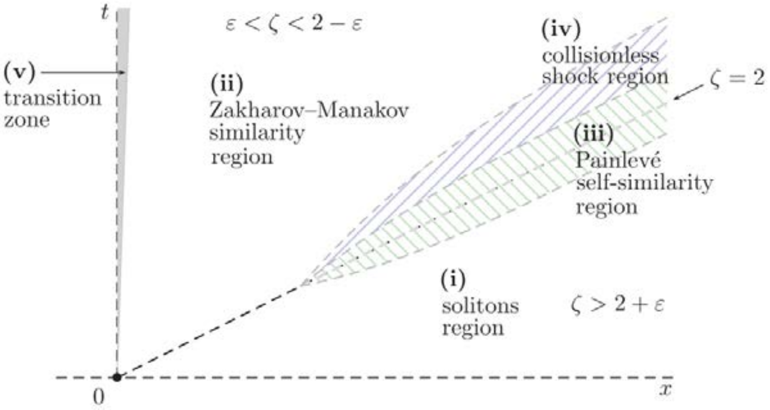
Open problem

- Computation of second Painlevé region.
- Boutet de Monvel et al. don not consider the "collisionless shock" region

$$C^{-1} < \left(2 - \frac{x}{t}\right) \left(\frac{t}{\log t}\right)^{\frac{2}{3}} < C, \;\; C > 1.$$

  (which occurs when $R(0) = -1$) present between the 1st Painlevé region and the 1st oscillatory region.
  Thisi is our case!

- Boutet de Monvel and Shepelsky (Ann. Inst. Fourier (Grenoble), 2009):



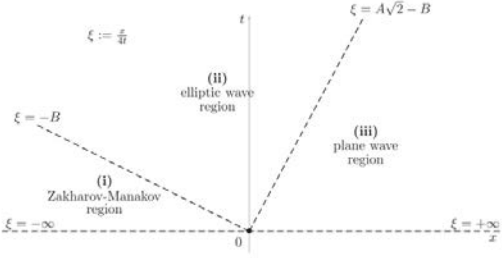
- Focusing nonlinear Schrödinger equation:

$$i\varepsilon\Psi_t + \frac{\varepsilon^2}{2}\Psi_{xx} + |\Psi|^2\Psi = 0, x \in \mathbb{R}, t > 0.$$

$$q(x,0) = q_0(x) = \begin{cases} 0, & x \leq 0, \\ Ae^{-2iBx}, & x > 0. \end{cases}$$

where $A > 0$, $B \in \mathbb{R}$: constants.

- Boutet de Monvel, Kotlyarov and Shepelsky (IMRN 2011): 3 regions of its long time asymptotics:



- Painlevé transcendents ?

Conclusion

- We consider the direct scattering analysis of the Camassa-Holm equation with a specified initial condition.
- Both of the continuous and discrete spectrum cases and the scattering data for the initial condition are derived explicitly.
- The reflection coefficient is non zero.
- The numerical simulation directly from CH equation looks match the results by Boutet de Monvel (2009), further numerical computations such as the Painlevé region and collisionless shock region are needed.

Thanks for Your Attention.